The Right Stuff: Appropriate Mathematics for All Students

Promoting the use of materials that engage students in meaningful activities that promote the effective use of technology to support mathematics, that further equip students with stronger problem solving and critical thinking skills, and enhance numeracy.

Overview

Students will apply the concepts of

- Functions – Students will be able to explain changes in the dependent variable as a result of changes to the independent variable.
- Graphs – Students will be able to explain how a graph changes when constants in the function change.

Students will be able to use technology to investigate relationships between variables and then explain those changes using mathematical terminology within the context of the problem.

The six scenarios within this module may be presented independently. However, notice the types of questions present in each scenario. The questions motivate discussions that deepen understanding.

Topics covered include: power functions, rational function, exponential, quadratic, and logistic; as well as a scenario that involves binomial probability.

Supplies and Materials

- 11.1 Student Worksheet
- Either 11.3 Excel files, 11.4 TI-Nspire files, or a handheld that will create a scatter plot and find a model for the data

Pre-requisite Knowledge

Students must be able copy data from Word into Excel or into a handheld, create a scatter plot, and find an appropriate algebraic model for the data.

Instructional Suggestions

This collection of problems is intended to provide examples of good questions to ask while examining problems within a real world context. Technology should provide students with insight into the relationship between variables controlled by parameters within the function. However, students must be able to analyze those relationships and explain them within the context of the problem.

#1 Real Estate Values – Appreciation and Depreciation
#2 Sharing the Bonus – Amount of Bonus Money and Number of Employees Sharing the Bonus
#3 Baked Potato – Temperature of Potato Over Time
#4 Rolling the Dice – Finding Maximum Probability
#5 Height of a Projectile – Analyzing Height vs Time and Finding Average Change in Height over Time
#6 Deer Population– Analyze a Population of Deer

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What Happens If…

#1 Real Estate Values: Appreciation and Depreciation

Changes to the value of real estate are normal. Historically, real estate appreciates in value over time. However, in the recent past, real estate has also depreciated. When real estate appreciates, the future value of the property is greater than the present value. The future value of real estate will be less if the property depreciates. The rate of change fluctuates, but can be averaged to estimate future worth.

The formula $V(years) = P(1 + APR)^{year}$ shows the future value, ***V***, as a function of the present value, ***P***, the amount of time in years, ***years***, and the average annual percentage rate of change (as a decimal), ***APR***. If we keep ***P*** and ***APR*** constant, this function is a *power function*.

In this activity, we will see how the future value ***V*** changes as we change the value of ***APR***?

Type a response to questions #1-5 below without using a graph. When your responses are complete, use the sliders and the graph to verify your response. Make changes as necessary to correct your statements.

1. What will the graph look like when ***APR*** = 0? Explain what this means within the context of the problem.
   Horizontal line. The value does not change value over time.

2. What will the graph look like when ***APR*** > 0? Explain what this means within the context of the problem.
   The graph is increasing and concave up. This means that the rate at which the future value is increasing (in dollars per year) becomes greater as ***c*** increases.

3. What will the graph look like when ***APR*** < 0? Explain what this means within the context of the problem.
   The graph is decreasing and concave up. This means that the rate at which the future value is decreasing (in dollars per year) becomes smaller as **c** increases.
   .

4. How does the future value change if we hold ***APR*** constant, ***APR*** = .02, but change ***P*** from $50,000 to $60,000?
   The graph will have a similar shape but will be shifted upwards, the y-intercept will increase by $10000 but the rate of increase will be greater causing the value after 25 years to have increased by $16,406.

5. Real estate currently worth $80,000 is expected to appreciate at an average annual rate of 2 ¹/₂ % over the next 10 years. What is the expected future value in ten years?
   $A = 80{,}000\ (1.025)^{10} = \$102{,}407$

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What Happens If…

#2 Sharing the Bonus

A company decided to reward those employees whose performance was exemplary. It created some predetermined qualifications and published them in the weekly newsletter. At the end of the year, they expected that several employees would qualify. Each qualifying employee would get a Performance Bonus consisting of a fixed amount plus an equal share of a large bonus. The formula **P(n) = e +** $^{b}/_{n}$ shows the performance bonus for each employee, ***P***, as a function of the number of qualifying employees, ***n,*** the fixed amount, ***e,*** and the amount of the large bonus, ***b***. In this activity, we will see how the performance bonus ***P*** changes if we keep ***e*** and ***b*** constant and change the value of ***n***?

Type a response to questions #1-4 below without using a graph. When your responses are complete, use the sliders and the graph to verify your response. Make changes as necessary to correct your statements. Use the graph to find your response to #5.

1. If ***b*** = 0, what is the shape of the graph as ***n*** increases? Explain what this means within the context of the problem.

   If ***b* = 0,** the equation becomes *P*(*n*) = e. As ***n*** changes the function value remains constant. This means that no matter how many employees the company has, everyone gets the same amount: e.

2. If the company changed ***e*** from \$100 to \$200, how would the graph change? Explain what this means within the context of the problem.

   The graph would shift upward by 100. This means that each employee would get an additional \$100 in their performance bonus.

3. What will the graph look like when ***e*** = 0? Explain what this means within the context of the problem.

   The graph will be concave up and decreasing. This means that as the number of people who are sharing the bonus ***b*** increases, the portion of ***b*** that each person gets decreases. However, the more people there are the less dramatic the drop that results from adding an additional person becomes.

4. How will the graph change if the company changed ***b*** from \$2500 to \$5000? Explain what this means within the context of the problem.

   The equation $P(n) = e + \dfrac{5000}{n}$ is equivalent to $P(n) = e + 2\left(\dfrac{2500}{n}\right)$. That is, this change doubles the amount of the large bonus that is being shared by all employees. The change is quite apparent for small values of n (because 2500/n is large and 2(2500/n) is larger).

5. The company decides to give an individual bonus \$300 to each qualifying employee and let them share a large bonus of \$5,000. How much will each of twenty qualifying employees get?

$$P(20) = 300 + \frac{5000}{20} = \$550$$

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What Happens If…

#3 Baked Potato

The temperature of a baking potato after *t* minutes in a heated oven is a function of several variables. Obviously, the temperature of the potato increases over time, but the rate of increase changes. The formula **$P(t) = v - (v - r)e^{-at}$** shows the temperature of the potato, ***P*** (measured from its center), as a function of the temperature of the oven, ***v***, the room temperature from which the potato was taken, ***r***, and the rate at which the potato heats, ***a***, which is a function of the size of the potato, among other things. How does the temperature of the potato change as time ***t*** (in minutes) increases?

How does the temperature of the potato change as ***t*** increases?

Type a response to questions #1-4 below without using a graph. When your responses are complete, use the sliders and the graph to verify your response. Make changes as necessary to correct your statements. Use the graph to find your response to #5.

1. What will the shape of the graph look like if ***v*** = ***r***? Explain what this means within the context of the problem.

   If ***v*** = ***r*** then the equation becomes P(t) = v (and you forgot to turn on the oven). The graph will be a horizontal line. This means that if the temperature of the oven is the same as room temperature the internal temperature of the potato will remain constant.

2. What will happen to the graph if ***v*** = 375, ***r*** = 65 and ***a*** is changed from 0.1 to 0.2? Explain what this means within the context of the problem.

   When ***v*** = 375, ***r*** = 65, and ***a*** = 0.1, the equation becomes $P(t) = 375 - (310)e^{-0.1t}$. When ***a*** = 0.2 the equation becomes $P(t) = 375 - (310)e^{-0.2t}$. The two functions have the same vertical intercept (0, 65) and are increasing towards a horizontal asymptote of $y = 375$. However, $-(310)e^{-0.2t}$ is increasing at a continuous rate of 20% while $-(310)e^{-0.1t}$ is increasing at a continuous rate of 10%. This means that the second graph is steeper than the first graph. In the context of the problem, this means the potato is probably smaller and it heats up more quickly. Thus, ***a*** is changed from 0.1 to 0.2.

3. As ***t*** gets large, beyond two hours, what is the shape of the graph? Explain what this means within the context of the problem.

   At ***t*** = 58 minutes, the temperature of the potato is 374.06 degrees. At ***t*** = 104 minutes, the temperature of the potato is 374.99 degrees. With a horizontal asymptote of 375.00 degrees, the temperature of the potato will essentially remain unchanged beyond two hours (***t* =** 120 minutes). Graphically speaking, this means the graph approximates the horizontal line ***y* =** 375 for values of ***t*** beyond ***t* =** 120 minutes.

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4. What is the vertical intercept of the graph? Explain what this means within the context of the problem.

   The vertical intercept is (0, 65). This means that initially the potato is at room temperature.

5. A potato is taken from a room at 65 degrees F and placed into an oven preheated to a temperature of 410 degrees F. What will the temperature of the potato be after 30 minutes if $\boldsymbol{a} = 0.2$?

   According to the model the temperature will be 409 degrees.

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What Happens If…

#4 Rolling The Dice

A die has six sides and is marked with a 1, 2, 3, 4, 5, or 6 on each face. Each face has an equally likely outcom interested in the number of times 6 is rolled. Use the Excel spreadsheet to perform the experiment of "rollir dice to see how often three sixes are rolled. A screen shot of the spreadsheet is shown below. Press F9 to re again). (In Excel, change the number of dice rolled by entering the desired number in the cell adjacent to the

You may roll between 3 and 20 dice.
How many dice do you want to roll? **12**

Die number:	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
Number shown on face:	3	4	4	1	3	3	2	4	3	1	6	2
Number of sixes:	**1**											

1. For how many dice is the probability of rolling exactly three sixes the greatest? Make a conjecture and believe what you do.

   The more dice we roll the greater probability there is that a 6 will appear. However, we want exactly 3 sixes. If we have be likely that we will end up with more than three sixes. If we have too few dice, it will be likely that we will end up witl sixes. We guess that somewhere between 12 and 20 dice will give the maximum probability of rolling exactly 3 dice.

2. According to binomial probability, the following function defines the probability of rolling exactly three

$$p(n) = \frac{n!}{3!(n-3)!}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^{n-3}$$
$$= \frac{n(n-1)(n-2)(5/6)^n}{750}$$

   Make a table of values for ***n*** = 3 to ***n*** = 20. What number of dice provides the greatest probability that tl rolled?

Number of dice rolled (n)	P(n)

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3	0.46%
4	1.54%
5	3.22%
6	5.36%
7	7.81%
8	10.42%
9	13.02%
10	15.50%
11	17.77%
12	19.74%
13	21.38%
14	22.68%
15	23.63%
16	24.23%
17	24.52%
18	24.52%
19	24.26%
20	23.79%

The maximum probability of 24.52% occurs when 17 or 18 dice are rolled.

3. What if you rolled ten dice and three sixes appeared on five such rolls in succession. Would you questi the dice?

Yes. The probability of rolling three sixes is 15.50% when ten dice are used. The probability of rolling exactly three sixes in $(0.1550)^5 = 0.00895\%$. Therefore, it is extremely unlikely that this will occur.

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What Happens If…

#5 Height of a Projectile

The height of a projectile is a function of the time according to **h(t) = 5 + 80t – 16t²** where ***h*** is the height of the projectile (in feet) and ***t*** is the time since launch (in seconds).

1. In the context of the problem, what is a reasonable domain for the function?
   t must be nonnegative since a negative time doesn't make sense in this context.
   We also know that ***h*** must be nonnegative since a negative height doesn't make sense. For values of ***t*** larger than 5.06 seconds, ***h*** is negative. So a reasonable domain is $0 \le t \le 5.06$.

2. Determine the maximum height of the projectile and the time it is reached.
   The maximum height will occur at the vertex.

$$\begin{aligned} x &= -\frac{b}{2a} \\ &= -\frac{80}{2(-16)} \\ &= 2.5 \text{ seconds} \end{aligned}$$

$$h(2.5) = 105 \text{ feet}$$

   The maximum height is 105 feet.

3. Find the change in the height between ***t*** = 0.5 and ***t*** = 0.7 second. Find the average change in height per second during this time period. Explain the results in the context of the problem.

$$\begin{aligned} \frac{h(0.7)-h(0.5)}{0.7-0.5} &= \frac{53.16-41}{0.2}\frac{\text{feet}}{\text{sec}} \\ &= \frac{12.16}{0.2}\frac{\text{feet}}{\text{sec}} \\ &= 60.8 \text{ feet per sec} \end{aligned}$$

   The height changed by 12.16 feet between 0.5 and 0.7 seconds. Over this time interval, the average rate of change was 60.8 feet per second. This means that the average upward speed was 60.8 feet per second.

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4. Find the change in the height between t = 0 and t = 1.2 second. Find the average change in height per second during this time period. Explain the result in the context of the problem.

$$\begin{aligned}\frac{h(1.2)-h(0)}{1.2-0} &= \frac{77.96-5}{1.2}\,\frac{\text{feet}}{\text{sec}}\\ &= \frac{72.96}{1.2}\,\frac{\text{feet}}{\text{sec}}\\ &= 60.8 \text{ feet per sec}\end{aligned}$$

The height changed by 72.96 feet between 0.0 and 1.2 seconds. Over this time interval, the average rate of change was 60.8 feet per second. This means that the average upward speed was 60.8 feet per second.

5. Find the change in the height between t = 1.1 and t = 1.3 seconds. Find the average change in height per second during this time period. Explain the result in the context of the problem.

$$\begin{aligned}\frac{h(1.3)-h(1.1)}{1.3-1.1} &= \frac{81.96-73.64}{0.2}\,\frac{\text{feet}}{\text{second}}\\ &= \frac{8.32}{0.2}\,\frac{\text{feet}}{\text{second}}\\ &= 41.6 \text{ feet per second}\end{aligned}$$

The height changed by 8.32 feet between 1.1 and 1.3 seconds. Over this time interval, the average rate of change was 41.6 feet per second. This means that the average upward speed was 41.6 feet per second.

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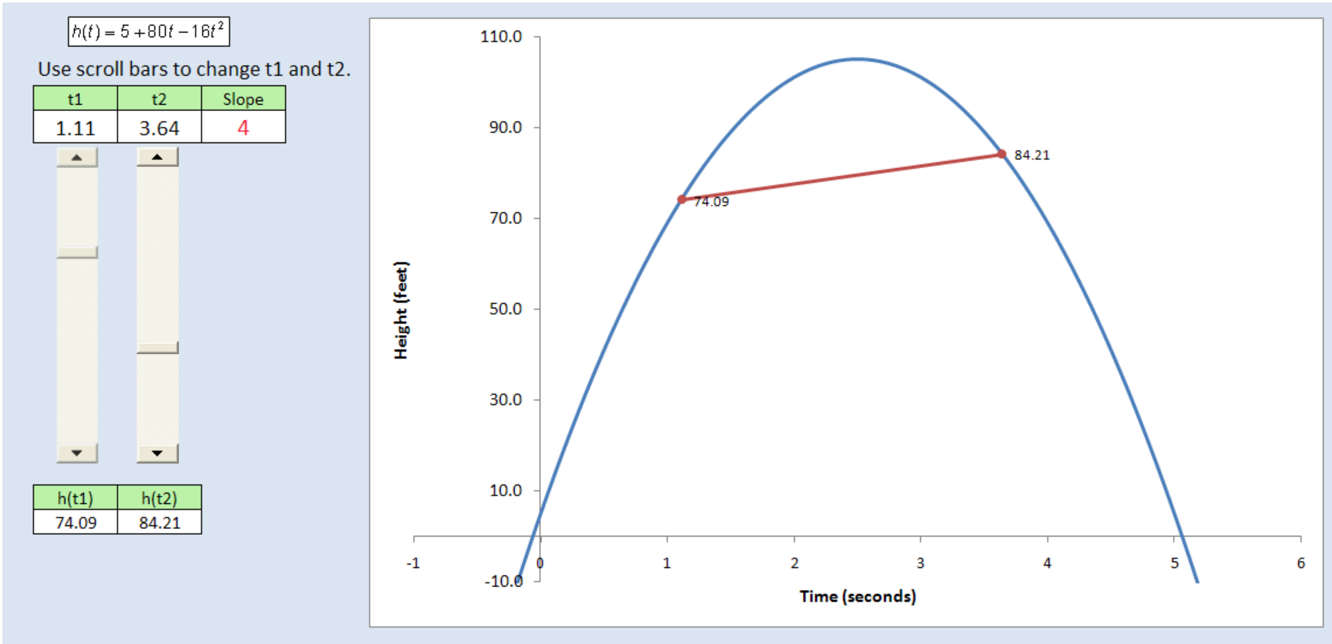
6. The sliders control the end points of the secant line. Adjust the secant line to show the average rate of change of height between the two times using the values in (c), (d), and (e). Can you adjust the secant so that the slope is -2? 2? 0?



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What Happens If…

#6 Deer Population

The number of deer within a state park has grown over the years to the extent that they have become dangerous to motorists along the roads that border the park. A study was done to determine the growth in population of the deer within the 580 acres of the park. The formula that best models the population is $P(t) = \frac{a}{b + c^t}$ where ***P*** is the deer population ***t*** years after 1990, ***a*** = 300, ***b*** = 0 .15 and ***c*** =0.8.

1. Find ***P***(0) and ***P***(20) and explain what these numbers mean in relation to the scenario.

$$P(0) = \frac{300}{0.15 + (0.8)^0} \approx 261 \qquad P(20) = \frac{300}{0.15 + (0.8)^{20}} \approx 1857$$

In 1990, there were 261 deer in the park. In 2010, there will be 1857 deer in the park.

2. About how many acres per deer were there at **t** = 0?

$$\frac{580 \text{ acres}}{261 \text{ deer}} \approx 2.22 \text{ acres per deer}$$

3. How may deer per acre were there at **t** = 20?

$$\frac{1857 \text{ deer}}{580 \text{ acres}} \approx 3.2 \text{ deer per acre}$$

4. Using Excel and the given values for **a**, **b**, and **c**, construct a table of values for **P(t)** for **t** = 1, 2, 3, … 20. As you construct the table, define the variables a, b, and c in the formula by anchoring them to adjacent cells that can easily be changed. (An absolute reference (anchor) relates a variable within a formula to a cell that does not change as the formula is copied. The absolute reference is accomplished with the use of "$".)

t	P(t)	t	P(t)
0	261	11	1272
1	316	12	1372
2	380	13	1464
3	453	14	1547
4	536	15	1620

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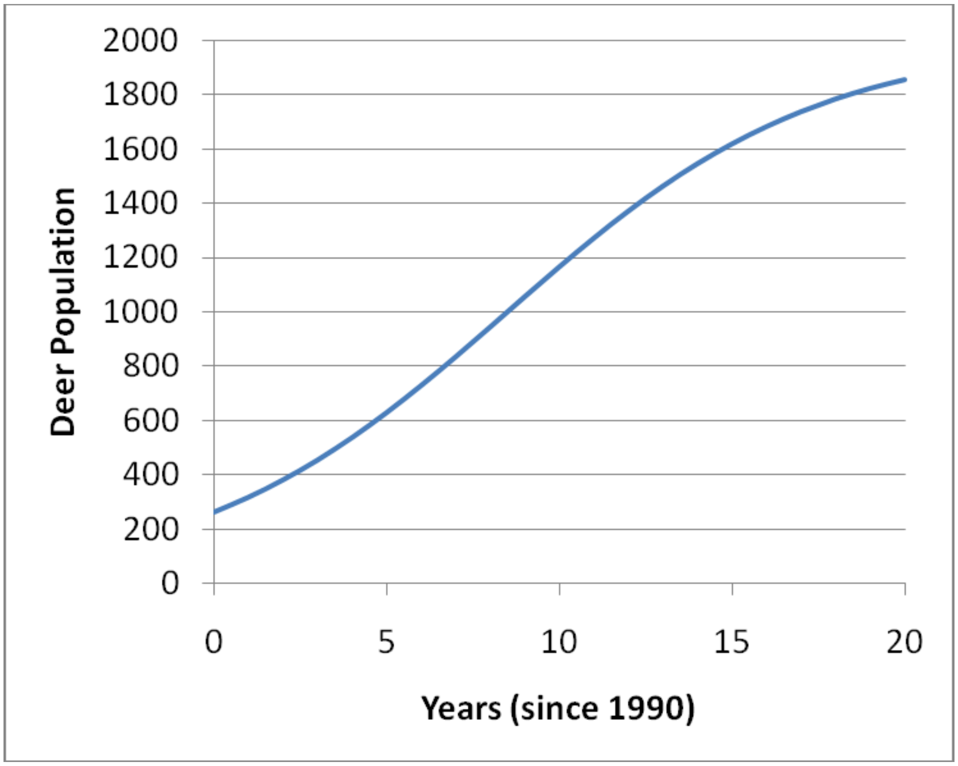
5	628	16	1684
6	728	17	1739
7	834	18	1786
8	944	19	1825
9	1056	20	1857
10	1166		

5. Construct a graph of the data.



6. Explain the shape of the curve within the context of the problem.

The graph is increasing and changes from concave up to concave down around 9 years. This means that the population growth rate was increasing up until about 1999. Between 1999 and 2010 the population continued to grow but the rate of growth slowed down.

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7. Change the value of ***a*** to 200 and explain what happens to the curve and what that means within the context of the problem.

The original function is $P(t)=\dfrac{300}{0.15+(0.8)^t}$. We'll name the new function N(t).

$$N(t)=\frac{200}{0.15+(0.8)^t}$$

$$=\frac{\frac{2}{3}(300)}{0.15+(0.8)^t} \quad \bullet \text{ Since } 200=\frac{2}{3}(300)$$

$$=\frac{2}{3}(P(t))$$

This graph of this new function in the graph of ***P*** vertically compressed by a factor of $\dfrac{2}{3}$. This means that the new population value is two-thirds of the old population value.

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